On Ga-threefolds whose algebraic quotient morphism only degenerates over isolated points

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ON $\mathbb{G}_a$-THREEFOLDS WHOSE ALGEBRAIC QUOTIENT MORPHISM ONLY DEGENERATES OVER ISOLATED POINTS

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Abstract. We study three dimensional complex normal affine varieties with an action of the additive group $\mathbb{G}_a$ which satisfy a special property which is defined in terms of the corresponding algebraic quotient morphism. This note is an expanded version of a talk given at the Kinosaki Algebraic Geometry Symposium in 2017 on a joint work with Adrien Dubouloz and Takashi Kishimoto.

A normal variety which has a nontrivial group action of the additive group $\mathbb{G}_a$ over the ground field, which we will assume to be the field of complex numbers, is called a $\mathbb{G}_a$-variety. Unlike the multiplicative group $\mathbb{G}_m$, which is the only other connected 1-dimensional affine algebraic group, the group $\mathbb{G}_a$ is not reductive. This means in particular that given an affine $\mathbb{G}_a$-variety $X = \operatorname{Spec}(A)$, we cannot in general expect the invariant ring $A^{\mathbb{G}_a} \subset A$ to be finitely generated so that a well defined quotient variety $X/\!/\mathbb{G}_a := \operatorname{Spec}(A^{\mathbb{G}_a})$ exists. However, due to a result of Zariski, $A^{\mathbb{G}_a}$ is finitely generated in case $\dim X \leq 3$ [3, p.45] and then we get a morphism of affine varieties $\rho\colon X \to X/\!/\mathbb{G}_a$ which is induced by the inclusion $A^{\mathbb{G}_a} \hookrightarrow A$.

The only affine $\mathbb{G}_a$-curve is $\mathbb{A}^1$, on which $\mathbb{G}_a$ acts by translations. Affine $\mathbb{G}_a$-surfaces have been classified by K.-H. Fieseler [2]. In the present note we study certain natural classes of affine $\mathbb{G}_a$-threefolds.

1. Additive actions on affine varieties

We recall some well known facts about $\mathbb{G}_a$-actions on affine normal varieties.

1.1. **Locally nilpotent derivations.** There is a natural one-to-one correspondence between $\mathbb{G}_a$-actions on an affine variety $X = \operatorname{Spec}(A)$ and locally nilpotent derivations on A.

Definition 1.1. A *locally nilpotent derivation* $D\colon A \to A$ is a $\mathbb{C}$-linear map which satisfies $D(fg) = fD(g) + gD(f)$ for all $f, g \in A$ and such that for every $f \in A$ there exists a natural number n (which may depned on f) such that $D^n(f) = 0$.

The correspondence between $\mathbb{G}_a$-actions on X and locally nilpotent derivations on A goes as follows: the comorphism of a $\mathbb{G}_a$-action $\alpha\colon \mathbb{G}_a \times X \to X$, $(t, x) \mapsto t * x$ has the form

$$\alpha^*\colon A \to A[T],\ f \mapsto \sum_{n=0}^{\infty} \frac{D^n f}{n!} T^n,$$

for the induced locally nilpotent derivation D. Note that $\ker D = A^{\mathbb{G}_a}$.

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Example 1.2. Let X be the affine variety given by $(xz = y^2 - 1) \subset \mathbb{A}^3_{x,y,z}$, and consider the $\mathbb{G}_a$-action on X which is given by the following locally nilpotent derivation:

$$D(x) = 0,\ D(y) = x,\ D(z) = 2y - 1.$$

In coordinates the action is given by $(x, y, z) * t = (x, y + xt, z + (2y - 1)t + xt^2)$ for $t \in \mathbb{G}_a$. The ring of invariant functions is given by $\mathbb{C}[x]$ and the quotient map is given by $\rho \colon X \to \mathbb{A}^1$, $(x, y, z) \mapsto x$. With $U := \mathbb{A}^1 \setminus \{o\}$, the complement of the origin, we have a $\mathbb{G}_a$-equivariant trivialisation $\rho^{-1}(U) \simeq U \times \mathbb{G}_a$ given by $(x, y, z) \mapsto (x, y/x)$, but $\rho^{-1}(o) = V(y^2 - 1) \subset X$ is the disjoint union of two affine lines, so the quotient morphism is not a principal $\mathbb{G}_a$-bundle, despite the $\mathbb{G}_a$-action on X being free.

1.2. **Trivialisations of the quotient morphism.** Example 1.2 shows that for an affine $\mathbb{G}_a$-variety X, there is in general no cover of $X/\!\!/\mathbb{G}_a$ by open subsets $U_i \subset X/\!\!/\mathbb{G}_a$ which have $\mathbb{G}_a$-equivariant trivialisations $\rho^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{G}_a$ – even when the given $\mathbb{G}_a$-action is free. However, we can always find at least one trivialisation of the quotient morphism.

Proposition 1.3. *Let $X = \mathrm{Spec}(A)$ be an affine $\mathbb{G}_a$-variety with finitely generated invariant ring so that the algebraic quotient exists as an affine variety; then there exists an open subset $U \subset X/\!\!/\mathbb{G}_a$ and an equivariant trivialisation $\rho^{-1}(U) \xrightarrow{\sim} U \times \mathbb{G}_a$, where $\mathbb{G}_a$ acts by translation on the second factor.*

Proof. Since the $\mathbb{G}_a$-action is nontrivial, we can find an element $f \in \ker D^2 \setminus \ker D$, where $D \colon A \to A$ denotes the corresponding locally nilpotent derivation. This means in particular that $f(x*t) = f(x) + Df(x)t$ for all $x \in X$ and for all $t \in \mathbb{G}_a$. Let $X^* := X_{Df}$ be the principal open subset of X where Df doesn't vanish, and let

$$S := X^* \cap \{x \in X \mid f(x) = 0\}.$$

Then

$$\begin{array}{rcl} S \times \mathbb{G}_a & \xrightarrow{\sim} & X^* \\ (x, t) & \mapsto & x * t \end{array}$$

is a bijective morphism and even an isomorphism by Zariski's Main Theorem since X^* is normal. Take $U := \rho(S \times \mathbb{G}_a) = (X/\!\!/\mathbb{G}_a)_{Df}$ to be the principal open subset where $Df \in A^{\mathbb{G}_a}$ doesn't vanish. □

Remark 1.4. A *slice* is an element $s \in A$ with $D(s) = 1$. If a slice exists, we have $A^{\mathbb{G}_a}[s] = A$. In the proof of Proposition 1.3, f/Df is a slice for the induced $\mathbb{G}_a$-action on the (invariant) open subset $X^* \subset X$, and hence we get $A_{Df} = A^{\mathbb{G}_a}_{Df}[\frac{f}{Df}]$. We call f/Df a *local slice*, since it is a slice which is defined only on a principal open subset of X. Giving a local slice is the same thing as giving a local trivialisation of the quotient morphism.

2. $\mathbb{G}_a$-extensions

Given a $\mathbb{G}_a$-variety X with finitely generated invariant ring $A^{\mathbb{G}_a}$, Proposition 1.3 shows that there is at least one open subset $U \subset X/\!\!/\mathbb{G}_a$ which has a $\mathbb{G}_a$-equivariant trivialisation. Now let $V \subset X/\!\!/\mathbb{G}_a$ be the union of all such open subsets. Then

$$\pi := \rho_{|\rho^{-1}(V)} \colon \rho^{-1}(V) \to V$$

is a principal $\mathbb{G}_a$-bundle.

$$\begin{tikzcd} \rho^{-1}(V) \arrow[r, hook] \arrow[d, "\pi"'] & X \arrow[d, "\rho"] \\ V \arrow[r, hook] & X/\!\!/\mathbb{G}_a \end{tikzcd}$$

Our aim is to study affine $\mathbb{G}_a$-threefolds with the special property that $(X/\!\!/\mathbb{G}_a) \setminus V$ consists only of isolated points. For simplicity of exposition, we will assume that $X/\!\!/\mathbb{G}_a \simeq \mathbb{A}^2$ and that $V = \mathbb{A}^2_* := \mathbb{A}^2 \setminus \{o\}$; see [1] for the more general setting where the same kind of $\mathbb{G}_a$-threefolds are studied for more general surfaces – in particular local 2-dimensional smooth schemes S and $V = S \setminus \{o\}$ the complement of the closed point. Changing the point of view slightly, this special property leads us to the notion of $\mathbb{G}_a$-extensions.

Definition 2.1. Let $\pi\colon P \to \mathbb{A}^n_*$ be a principal $\mathbb{G}_a$-bundle. A $\mathbb{G}_a$-*extension* of P is a morphism $\hat{\pi}\colon \hat{P} \to \mathbb{A}^n$ which fits in the following commutative diagram, where $\hat{P}$ is a $\mathbb{G}_a$-variety, $P \hookrightarrow \hat{P}$ is a $\mathbb{G}_a$-equivariant open embedding and $\hat{\pi}^{-1}(\mathbb{A}^n_*) = \iota(P)$.

$$\begin{tikzcd} P \arrow[r, hook, "\iota"] \arrow[d, "\pi"'] & \hat{P} \arrow[d, "\hat{\pi}"] \\ \mathbb{A}^n_* \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$

Thus $\hat{P}$ is obtained from P by adding a fibre over the origin $o \in \mathbb{A}^n$, which we will refer to as the *exceptional fibre* of the $\mathbb{G}_a$-extension. Affine $\mathbb{G}_a$-extensions, i.e. $\mathbb{G}_a$-extensions with $\hat{P}$ affine, will be of particular interest.

Since $\mathbb{A}^1_*$ is affine, the only principal $\mathbb{G}_a$-bundle on $\mathbb{A}^1_*$ is the trivial bundle $P \simeq \mathbb{A}^1_* \times \mathbb{G}_a$. A complete classification of $\mathbb{G}_a$-extensions for $n = 1$ is given in [2].

2.1. A basic example of an affine $\mathbb{G}_a$-extension.

For $n = 2$ we can take $P = \mathrm{SL}_2 \simeq (xv - yu = 1) \subset \mathbb{A}^4_{x,y,u,v}$; this is a $\mathbb{G}_a$-variety as follows

$$\begin{array}{rcl} \mathrm{SL}_2 \times \mathbb{G}_a & \to & \mathrm{SL}_2, \\ \left(\begin{bmatrix} x & u \\ y & v \end{bmatrix}, t \right) & \mapsto & \begin{bmatrix} x & u \\ y & v \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & u + tx \\ y & v + ty \end{bmatrix}. \end{array}$$

The corresponding locally nilpotent derivation is given by $D(x) = D(y) = 0$, $D(u) = x$, $D(v) = y$. We have $\ker D = \mathbb{C}[x, y]$, so $\mathrm{SL}_2/\!\!/\mathbb{G}_a = \mathbb{A}^2$. Note that the quotient morphism $\mathrm{SL}_2 \to \mathbb{A}^2$, $(x, y, u, v) \mapsto (x, y)$ is not surjective: the image is $\mathbb{A}^2_*$. There are trivialisations over the sets $(x \neq 0)$ and $(y \neq 0)$, given by the local slices $g_x = u/x$ and $g_y = v/y$ respectively. Dividing the relation $xv - yu = 1$ by xy, we get $g_y - g_x = \frac{1}{xy}$. The $\mathbb{G}_a$-principal bundles on $\mathbb{A}^2_*$ are classified up to isomorphism by the cohomology group $H^1(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*}) \simeq x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}]$ and it can be argued that SL_2 is the simplest possible among nontrivial principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$ since it corresponds to the cocycle $\frac{1}{xy} \in H^1(\mathbb{A}^2_*, \mathcal{O}_{\mathbb{A}^2_*})$.

Example 2.2. We construct an affine $\mathbb{G}_a$-extension X_0 of SL_2, and for this we consider the $\mathbb{G}_m$-action on SL_2 which is given by

$$\begin{array}{rcl} \mathrm{SL}_2 \times \mathbb{G}_m & \to & \mathrm{SL}_2, \\ \left(\begin{bmatrix} x & u \\ y & v \end{bmatrix}, \lambda \right) & \mapsto & \begin{bmatrix} \lambda x & \lambda^{-1}u \\ \lambda y & \lambda^{-1}v \end{bmatrix}. \end{array}$$

The $\mathbb{G}_m$-quotient is given by $\mathrm{SL}_2 \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, $\begin{bmatrix} x & u \\ y & v \end{bmatrix} \mapsto ([x:y],[u:v])$, and we take

$$X_0 := \mathrm{SL}_2 \times^{\mathbb{G}_m} \mathbb{A}^1 := (\mathrm{SL}_2 \times \mathbb{A}^1)/\mathbb{G}_m$$

to be the twisted product of SL_2 and $\mathbb{A}^1$, where the $\mathbb{G}_m$ acts as follows:

$$\begin{array}{rcl} \mathbb{G}_m \times (\mathrm{SL}_2 \times \mathbb{A}^1) & \to & (\mathrm{SL}_2 \times \mathbb{A}^1), \\ (\lambda, (x,y)) & \mapsto & (x\lambda, \lambda^{-1} y). \end{array}$$

Thus X_0 is obtained from SL_2 by adding in a zero section to the principal $\mathbb{G}_m$-bundle $\mathrm{SL}_2 \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ so that the general fibre $\mathbb{G}_m$ is replaced by $\mathbb{A}^1$. It follows that X_0 is a line bundle over the affine surface $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, so in particular X_0 is affine. The following map is an open embedding of SL_2 into X_0, which in turn embeds as the vanishing locus of three 2×2-minors in $\mathbb{A}^5$:

$$\begin{array}{rcl} \mathrm{SL}_2 & \hookrightarrow & X_0 \simeq \left(\bigwedge^2 \begin{bmatrix} x & q & p \\ y & r & q-1 \end{bmatrix} = 0 \right) \subset \mathbb{A}^5_{x,y,p,q,r}, \\ \begin{bmatrix} x & u \\ y & v \end{bmatrix} & \mapsto & (x, y, xu, xv, yv). \end{array}$$

The $\mathbb{G}_a$-action on SL_2 induces a $\mathbb{G}_a$-action on X_0 given by $D(x) = D(y) = 0$, $D(p) = x^2$, $D(q) = xy$, $D(r) = y^2$.

Remark 2.3. In Example 2.2 we have

$$\begin{array}{rcl} X_0 \setminus \mathrm{SL}_2 & = & \hat{\pi}^{-1}(o) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \simeq (q(q-1) = pr) \subset \mathbb{A}^3_{p,q,r} \\ & = & X_0^{\mathbb{G}_a}; \end{array}$$

in particular the induced $\mathbb{G}_a$-action on X_0 is not free. Note that the exceptional fibre $\hat{\pi}^{-1}(o)$ is uniruled.

2.2. **Uniruledness of the exceptional fibre.** The last part of Remark 2.3 holds in general for smooth $\mathbb{G}_a$-extensions of nontrivial $\mathbb{G}_a$-bundles. We say that a variety X is $\mathbb{A}^1$-*uniruled* if the set

$$\{x \in X \mid \exists h\colon \mathbb{A}^1 \to X \text{ non-constant, such that } x \in h(\mathbb{A}^1)\}$$

is dense in X.

Proposition 2.4 ([1])**.** *Let $\hat{\pi}\colon \hat{P} \to \mathbb{A}^2$ be a smooth affine $\mathbb{G}_a$-extension of a nontrivial principal $\mathbb{G}_a$-bundle $\pi\colon P \to \mathbb{A}^2_*$. Then every irreducible component of $\hat{\pi}^{-1}(o)_{\mathrm{red}}$ is $\mathbb{A}^1$-uniruled.*

2.3. **Principal $\mathbb{G}_a$-bundles and line bundles vs. locally trivial $\mathbb{A}^1$-bundles.** The map $\hat{\pi}\colon X_0 \to \mathbb{A}^2$ restricts to a principal $\mathbb{G}_a$-bundle on $\hat{\pi}^{-1}(\mathbb{A}^2_*) \simeq \mathrm{SL}_2 \subset X_0$, so in particular all fibres over $\mathbb{A}^2_*$ are isomorphic to $\mathbb{G}_a$, but the fibre $\hat{\pi}^{-1}(o)$ is a surface. A remarkable property of $\hat{\pi}\colon X_0 \to \mathbb{A}^2$ is that it factors through a locally trivial $\mathbb{A}^1$-bundle $\hat{\pi}'\colon X_0 \to \widetilde{\mathbb{A}^2}$ over the blowup $p\colon \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ of $o \in \mathbb{A}^2$. The $\mathbb{A}^1$-bundles are natural generalisations of principal $\mathbb{G}_a$-bundles. With $\hat{\pi}'\colon X_0 \to \widetilde{\mathbb{A}^2}$ defined as

$$\hat{\pi}'\colon (x,y,p,q,r) \mapsto ((x,y),[x:y]) = ((x,y),[q:r]) = ((x,y),[p:q-1]),$$

we have $\hat{\pi} = p \circ \hat{\pi}'$. Note that $\hat{\pi}'$ has trivialisations over $(x \neq 0)$ and $(y \neq 0)$ respectvely. It follows from Theorem 1 below that this factorisation property determines X_0 uniquely among affine $\mathbb{G}_a$-extensions of SL_2.

Let $\theta\colon W \to B$ be a locally trivial $\mathbb{A}^1$-bundle so that, by definition, there is an open cover $B = \cup_{i \in I} U_i$ of B, trivialisations $\theta^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^1$ and transition functions on the intersections $U_i \cap U_j$ given by $U_j \times \mathbb{A}^1 \to U_i \times \mathbb{A}^1$, $(x, v) \mapsto (x, a_{ij}v + b_{ij})$, where the $a_{ij} \in H^1(B, \mathcal{O}_B^*)$ form the cocycle of a line bundle on B and the $b_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfy $b_{ik} = a_{ij}b_{jk} + b_{ij}$ over any triple intersection $U_i \cap U_j \cap U_k$.

Definition 2.5. Given a locally trivial $\mathbb{A}^1$-bundle $\theta\colon W \to B$ as above, *the line bundle associated to* W is the line bundle $L_W \to B$ which is given by the cocycle $a_{ij} \in H^1(B, \mathcal{O}_B^*)$. We denote by $\mathcal{L}_W$ the invertible sheaf on B of sections in L_W.

Proposition 2.6. *There is a natural one-to-one correspondence between θ-invariant $\mathbb{G}_a$-actions on the total space of a locally trivial $\mathbb{A}^1$-bundle $\theta\colon W \to B$ and $\Gamma(B, \mathcal{L}_W)$.*

Proof. For a global section $\{(U_i, s_i)\}_{i\in I} \in \Gamma(B, \mathcal{L}_W)$, we define a $\mathbb{G}_a$-action locally by $(x, v) * t := (x, v + s_i(x)t)$ over U_i. Acting on $(x, v) \in U_j \times \mathbb{A}^1$ by $t \in \mathbb{G}_a$ and then changing to the chart U_i takes us to $(x, a_{ij}v + a_{ij}s_j(x)t + b_{ij})$ whereas first changing to the chart U_i and then acting by $t \in \mathbb{G}_a$ takes to $(x, a_{ij}v + s_i(x)t + b_{ij})$. Hence the $\mathbb{G}_a$-action is well defined. Conversely, a θ-invariant $\mathbb{G}_a$-action on W necessarily has the form $(x, v) \mapsto (x, v + s_i(x)t)$ over U_i for some $s_i \in \mathcal{O}(U_i)$ and it follows from the above calculation that $s_i = a_{ij}s_j$ over $U_i \cap U_j$ so that the s_i glue to a section in $\Gamma(B, \mathcal{L}_W)$. □

3. Locally trivial $\mathbb{A}^1$-bundles on the blowup $\widetilde{\mathbb{A}^2}$ of the origin in $\mathbb{A}^2$

Like X_0, some $\mathbb{G}_a$-extensions $\hat{\pi}\colon \hat{P} \to \mathbb{A}^2$ of a principal $\mathbb{G}_a$-bundle $\pi\colon P \to \mathbb{A}^2_*$ have the extra property of factoring through a locally trivial $\mathbb{A}^1$-bundle $\theta\colon W \to \widetilde{\mathbb{A}^2}$ over the blowup of the origin in $\mathbb{A}^2$; in [1] such extensions are called *of Type I.* Since $\operatorname{Pic}(\widetilde{\mathbb{A}^2}) = \mathbb{Z}$, the sections of the associated line bundle $L_W \to \widetilde{\mathbb{A}^2}$ are given by an invertible sheaf $\mathcal{O}_{\widetilde{\mathbb{A}^2}}(lE)$ for some $l \in \mathbb{Z}$. The following proposition gives a criterion for when an extension of Type I is affine.

Proposition 3.1. *Let $\theta\colon W \to \widetilde{\mathbb{A}^2}$ be a locally trivial $\mathbb{A}^1$-bundle. Then the following statements are equivalent, where $E \subset \widetilde{\mathbb{A}^2}$ denotes the exceptional divisor of the blowup.*

(1) W is affine.

(2) The induced map $W|_E \to E$ is a nontrivial locally trivial $\mathbb{A}^1$-bundle.

(3) The isomorphism class of W in $H^1(\widetilde{\mathbb{A}^2}, \mathcal{O}_{\widetilde{\mathbb{A}^2}}(lE))$ does not belong to the image of the natural inclusion $H^1(\widetilde{\mathbb{A}^2}, \mathcal{O}_{\widetilde{\mathbb{A}^2}}((l-1)E)) \hookrightarrow H^1(\widetilde{\mathbb{A}^2}, \mathcal{O}_{\widetilde{\mathbb{A}^2}}(lE))$.

The total space of a principal $\mathbb{G}_a$-bundle $\pi\colon P \to \mathbb{A}^2_*$ is affine if and only if π is a nontrivial bundle. With exception for the trivial $\mathbb{G}_a$-bundle on $\mathbb{A}^2_*$, every principal $\mathbb{G}_a$-bundle on $\mathbb{A}^2_*$ has a unique affine $\mathbb{G}_a$-extension which factors through a locally trivial $\mathbb{A}^1$-bundle on $\widetilde{\mathbb{A}^2}$.

Theorem 1 ([1])**.** *Let $\pi\colon P \to \mathbb{A}^2_*$ be a nontrivial principal $\mathbb{G}_a$-bundle. Then there exists a uniquely determined affine $\mathbb{G}_a$-extension $\hat{\pi}\colon W \to \mathbb{A}^2$ of P such that $\hat{\pi}$ factors through a locally trivial $\mathbb{A}^1$-bundle $\theta\colon W \to \widetilde{\mathbb{A}^2}$.*

$$\begin{tikzcd}
P \arrow[r, hook] \arrow[dd, "\pi"'] & W \arrow[d, "\theta"] \arrow[dd, bend left, "\hat{\pi}"] \\
& \widetilde{\mathbb{A}^2} \arrow[d] \\
\mathbb{A}^2_* \arrow[r, hook] & \mathbb{A}^2
\end{tikzcd}$$

Given a nontrivial principal $\mathbb{G}_a$-bundle $\pi\colon P \to \mathbb{A}^2_*$, let W be the unique affine extension of Type I which is obtained by Theorem 1. Using the correspondence between global sections in $\Gamma(B, \mathcal{L}_W)$ and θ-invariant $\mathbb{G}_a$-actions on W, we see that

- $l < 0$: cannot happen since $\mathcal{L}_W$ only has the zero section, and hence there would be no $\mathbb{G}_a$-actions on W, contradicting that W is a $\mathbb{G}_a$-variety.
- $l = 0 \Leftrightarrow L_W$ is trivial $\Leftrightarrow \exists$ non-vanishing section $\Leftrightarrow W$ has a free $\mathbb{G}_a$-action.
- $l > 0$: implies that there is a section which vanishes to the order l along E, so that W has a $\mathbb{G}_a$-action with fixed points of order l above E.

Example 3.2. For the affine $\mathbb{G}_a$-extension X_0 we have $l = 2$.

Given a principal $\mathbb{G}_a$-bundle $\pi\colon P \to \mathbb{A}^2_*$ as above, a well chosen affine modification of the associated affine $\mathbb{G}_a$-extension $W \to \mathbb{A}^2$ will yield a new affine $\mathbb{G}_a$-extension of the same principal $\mathbb{G}_a$-bundle with a lower fixed point order l. Iterated affine modifications will lead to affine $\mathbb{G}_a$-extensions of $\pi\colon P \to \mathbb{A}^2_*$ with free $\mathbb{G}_a$-action, which are themselves principal $\mathbb{G}_a$-bundles over a surface of the form $S_n(o_1, \ldots, o_{n-1})$ to be defined in Section 5. Roughly speaking, for $n \geq 2$, these surfaces are obtained by first blowing up $\mathbb{A}^2$ in $n-1$ successive points $o_1, \ldots, o_{n-1}$, each of which lies on the exceptional divisor of the previous blowup, and finally deleting all but the last exceptional divisor. The surface $S_1(o_1)$ is obtained by blowing up $\mathbb{A}^2$ in a point, and then deleting a point o_1 on the exceptional divisor.

4. Affine modifications

Let $X = \operatorname{Spec}(A)$ be an affine variety, $D = \operatorname{div}(f)$ a principal divisor on X and $I = (b_1, \ldots, b_s)$ an ideal containing f. The affine modification of X with respect to (f, I) is

$$X' := \operatorname{Spec}(A[b_1/f, \ldots, b_s/f]) \to X.$$

Remark 4.1. If D is irreducible with smooth support, then X' is just the complement of the strict transform of D in $\operatorname{Bl}_I(X)$.

Example 4.2. The following smooth variety

$$X_1 := \left(\bigwedge^2 \begin{bmatrix} x & y & z_1 \\ (yz_1 + 1) & w & z_2 \end{bmatrix} = 0 \right) \subset \mathbb{A}^5_{x,y,z_1,z_2,w},$$

given by the vanishing of three 2×2-minors in $\mathbb{A}^5$, is obtained by affine modification of X_0. A $\mathbb{G}_a$-equivariant embedding of SL_2 is given by

$$\begin{array}{ccc} \mathrm{SL}_2 & \hookrightarrow & X_1 \\ \begin{bmatrix} x & u \\ y & v \end{bmatrix} & \mapsto & (x, y, u, uv, yv), \end{array}$$

and the induced $\mathbb{G}_a$-action on X_1 is given by the locally nilpotent derivation $D(x) = D(y) = 0$, $D(z_1) = x$, $D(z_2) = 2yz_1 + 1$, $D(w) = y^2$. This is a free $\mathbb{G}_a$-action and the orbit space $X_1/\mathbb{G}_a$ is isomorphic to $\widetilde{\mathbb{A}^2} \setminus \{\text{one point on E}\} \simeq S_1(o_1)$. In fact, the induced $\mathbb{G}_a$-action is even proper.

Definition 4.3. A regular group action $G \times X \xrightarrow{\mu} X$ of an algebraic group G is called *proper* if

$$(\mu \times \mathrm{pr}_2)\colon G \times X \to X \times X$$

is a proper morphism.

The affine modification in Example 4.2 starts in X_0 with $l = 2$ and produces the affine $\mathbb{G}_a$-extension X_1 with $l = 0$. The exceptional fibre of X_1 is $\mathbb{A}^2$, which can be argued to be the simplest possible surface that can appear as exceptional fibre of a $\mathbb{G}_a$-extension. A smooth $\mathbb{G}_a$-extension with exceptional fibre $\mathbb{A}^2$ and whose induced $\mathbb{G}_a$-action is proper, is called *of Type II* in [1].

5. Applications

Since a proper $\mathbb{G}_a$-action is free, we can define the geometric quotient $X/\mathbb{G}_a$. It is an algebraic space in general, but a scheme in our case since $\dim X \leq 3$. The algebraic quotient morphism factors through the geometric quotient:

$$X \to X/\mathbb{G}_a \to X/\!\!/\mathbb{G}_a.$$

The geometric quotient morphism is only locally trivial with respect to the étale topology in general, but Zariski locally trivial in case $\dim X \leq 3$ and X is smooth. It is well known that a $\mathbb{G}_a$-action is proper if and only if $X/\mathbb{G}_a$ is separated.

Extensions of Type II can be produced as X_1 was produced from X_0, by performing suitably chosen affine modifciations of $\mathbb{G}_a$-extensions that factor through a locally free $\mathbb{A}^1$-bundle on $\widetilde{\mathbb{A}^2}$. This leads to $\mathbb{G}_a$-extensions of Type II with geometric quotient among the surfaces $S_n(o_1, \ldots, o_{n-1})$ which we now define. They depend on the choice of $n-1$ points $o_1, \ldots, o_{n-1}$.

Definition 5.1. Let $\overline{\tau}_1 \colon (\overline{S}_1, \overline{E}_1) \to (\mathbb{A}^2, o)$ be the blowup of the origin $o \in \mathbb{A}^2$, and

(1) $\overline{\tau}_2 \colon \overline{S}_2(o_1) \to \overline{S}_1$ the blowup of $o_1 \in \overline{E}_1$,

(2) for $2 \leq k \leq n-1$, $\overline{\tau}_{k+1} \colon \overline{S}_{k+1}(o_1, \ldots, o_k) \to \overline{S}_k(o_1, \ldots, o_{k-1})$ the blowup of $o_k \in \overline{E}_k$.

The points $o_i \in \overline{E}_i$ are chosen arbitrarily on the previous exceptional divisor $\overline{E}_i \simeq \mathbb{P}^1$ for $2 \leq k \leq n-2$, and for the last point we require that $o_{n-1} \in \overline{E}_{n-1}$ be chosen so that $\overline{E}_n$ intersects the tree of rational curves $(\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1})$ transversally in a unique point. Finally we define $S_1(o_1) := \overline{S}_1 \setminus \{o_1\}$, $E_1 := \overline{E}_1 \cap S_1 \simeq \mathbb{A}^1$, $S_n(o_1, \ldots, o_{n-1}) := \overline{S}_n(o_1, \ldots, o_{n-1}) \setminus (\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1})$, $E_n := S_n(o_1, \ldots, o_{n-1}) \cap \overline{E}_n \simeq \mathbb{A}^1$, $\tau_1 := \overline{\tau}_1|_{S_1(o_1)}$ and $\tau_i := \overline{\tau}_i|_{S_i(o_1, \ldots, o_{i-1})}$ for $2 \leq i \leq n-1$.

Theorem 2 ([1]). *Let $\pi \colon P \to \mathbb{A}^2_*$ be a nontrivial principal $\mathbb{G}_a$-bundle. Then for every $n \geq 1$ and for every surface $S_n(o_1, \ldots, o_{n-1})$, there is a principal $\mathbb{G}_a$-bundle $q \colon X \to S_n(o_1, \ldots, o_{n-1})$ such that the following diagram commutes, where X is an extension of Type II with $X/\mathbb{G}_a = S_n(o_1, \ldots, o_{n-1})$.*

$$\begin{tikzcd}
P \arrow[r, hook, "j"] \arrow[d, "\pi"'] & X \arrow[d, "q"] \\
S_n(o_1, \ldots, o_{n-1}) \setminus E_n \arrow[d, "\simeq"'] & S_n(o_1, \ldots, o_{n-1}) \arrow[d] \\
\mathbb{A}^2_* \arrow[r, hook] & \mathbb{A}^2
\end{tikzcd}$$

Theorem 3 ([1]). *Conversely, for any extension X of Type II of a nontrivial principal $\mathbb{G}_a$-bundle $\pi \colon P \to \mathbb{A}^2_*$, there exists $n \geq 1$ and a surface $S_n(o_1, \ldots, o_{n-1})$ such that $q \colon X \to S_n(o_1, \ldots, o_{n-1}) \simeq X/\mathbb{G}_a$ is a principal $\mathbb{G}_a$-bundle and $\pi \colon P \to \mathbb{A}^2_*$ coincides with $q|_{q^{-1}(S_n(o_1, \ldots, o_{n-1}) \setminus E_n)}$.*

Thus we have obtained a description of the extensions of Type II in terms of their geometric quotients. The geometric quotient of an extension of Type II is one of the surfaces $S_n(o_1, \ldots, o_{n-1})$ and conversely every surface $S_n(o_1, \ldots, o_{n-1})$ appear as the geometric quotient of an extension of Type II. Unfortunately we don't know whether the extensions of Type II are all affine, but at least we know that for each prescribed geometric quotient $S_n(o_1, \ldots, o_{n-1})$, there is an extension of Type II with that quotient and whose total space is affine.

Theorem 4 ([1])**.** *Let $\pi \colon P \to \mathbb{A}^2_*$ be a nontrivial principal $\mathbb{G}_a$-bundle. Then for every $n \geq 1$ and every surface $S_n(o_1, \ldots, o_{n-1})$, there exists an affine extension X of Type II with $X/\mathbb{G}_a = S_n(o_1, \ldots, o_{n-1})$.*

Question 5.2. *Is an extension of Type II automatically affine?*

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